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Random Walks on the Bethe Lattice

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We obtain random walk statistics for a nearest-neighbor (Pólya) walk on a Bethe lattice (infinite Cayley tree) of coordination number z, and show how a random walk problem for a particular inhomogeneous Bethe lattice may be solved exactly. We question the common assertion that the Bethe lattice is an infinite-dimensional system.

KEY WORDS: Random walks; Bethe lattice; Cayley tree; first passage times.

1. INTRODUCTION

The Cayley tree[1] is a well-defined topological entity, consisting of *sites*, connected by *bonds*, with the following properties:

(i) each site has the same coordination number z (i.e., each site is connected to z nearest neighbors);

(ii) no closed loops exist.

In the case $z = 2$, the Cayley tree is a one-dimensional object (a linear chain). For $z \geqslant 3$, a Cayley tree can be embedded in two-dimensional Euclidean space, but if the tree is of infinite extent the angles between bonds and the lengths of bonds cannot both be bounded below by any positive constant. Since an embedding with constant bond lengths and angles is not possible in any finite-dimensional Euclidean space, the Cayley tree of infinite extent is often described as a "pseudolattice of infinite dimension," and is often called the Bethe lattice.[2]

The Bethe lattice has a uniquely simple topological structure: it is simply connected, and all sites are topologically equivalent. This structure renders a number of otherwise very difficult problems of statistical physics

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analytically solvable. A useful short review of these matters has been given by Thorpe,[3] though many additional references may be added to his collection.[4] Although some work has been done on transport properties of Bethe lattices,[5] the present authors have not encountered any investigation of the statistical properties of random walks on Bethe lattices. Such an investigation is the objective of the present paper; particular emphasis is placed on the statistical properties of walks of long duration.

In Section 2, we extend the Montroll[6] generating function formalism for random walks on Bravais lattices to the Bethe lattice. Many existing calculations of random walk statistics for Bravais lattices[7] can consequently be modified for application to the Bethe lattice. In Section 3, we derive random walk statistics for a Pólya[8] (unbiased, nearest-neighbor) walk on a Bethe lattice, including the probability of a recurrent walk (one for which the walker returns to the starting point), the mean duration of a recurrent walk, the probability of reaching any given site and the mean first-passage time to this site, and the mean number of distinct sites visited in a walk of n steps as $n \to \infty$. The basic formalism developed here is not easily adapted to inhomogeneous Bethe lattices (for which the transition probabilities vary from site to site), but we show in Section 4 how the problem of a random walk on a particular inhomogeneous Bethe lattice (corresponding to a central force field) can be solved exactly. Finally, in Section 5, we compare random walk statistics for the Bethe lattice with those for Bravais lattices, and address the question of the subjective definition of the dimensionality of a Bethe lattice.

2. RANDOM WALK GENERATING FUNCTIONS

We select any given site of the Bethe lattice as origin of coordinates. Any other site on the lattice is connected to this origin by a unique path. If this path consists of l bonds, we assign to the site the coordinate l. (Hence there is a unique site, the origin, with $l = 0$; there are z sites with $l = 1$; and there are in general $z(z-1)^{l-1}$ sites with the same coordinate l.) Restricting our attention to unbiased, nearest-neighbor (Pólya[8]) stepping, we note that a step will either take the walker further from the origin, with probability $(z-1)/z$, or closer to the origin, with probability $1/z$. The walk may therefore be described as a biased, one-dimensional lattice walk, and the transition probability from a site with coordinate l' to a site with coordinate l is

$$\gamma(l, l') = \begin{cases} \left(1 - \frac{1}{z}\right)\delta_{l,l'+1} + \frac{1}{z}\,\delta_{l,l'-1}, & l' \geqslant 1 \\ \delta_{l,l'+1}, & l' = 0 \end{cases} \tag{1}$$

The origin acts as a reflecting barrier.

We consider a walk for which the starting point has coordinate m. (It is important here to distinguish between the starting point of the walk, called by many authors "the origin," and the origin of coordinates. In the present paper, the term "origin" is reserved for the origin of coordinates.) Let $P_n(l \mid m)$ denote the probability that the walker has coordinate l after n steps, so that the initial condition is

$$P_0(l \mid m) = \delta_{lm} \tag{2}$$

The evolution of the walk is governed by the equation

$$P_{n+1}(l \mid m) = \sum_{l'} \gamma(l, l') P_n(l' \mid m) \tag{3}$$

and the solution for $P_n(l \mid m)$ can be determined via generating functions by treating the origin as a defect in the one-dimensional lattice onto which the Bethe lattice has been mapped. Following Montroll's treatment of defective Bravais lattices,[7] we write

$$\gamma(l, l') = p(l - l') + q(l, l') \tag{4}$$

where

$$p(l) = \left(1 - \frac{1}{z}\right)\delta_{l,1} + \frac{1}{z}\delta_{l,-1} \tag{5}$$

and

$$q(l, l') = \begin{cases} 0, & l' \neq 0 \\ \dfrac{1}{z}\delta_{l,1} - \dfrac{1}{z}\delta_{l,-1}, & l' = 0 \end{cases} \tag{6}$$

Here the integers l, l' are allowed to take negative values. So long as the starting coordinate is non-negative, the "defect" at the origin guarantees that $P_n(l \mid m) = 0$ for $l < 0$.

It follows from Eqs. (3) and (4) that

$$P_{n+1}(l \mid m) - \sum_{l'} p(l - l') P_n(l' \mid m) = \sum_{l'} q(l, l') P_n(l' \mid m) \tag{7}$$

and introducing a generating function, defined for $|\xi| < 1$ by

$$P(l \mid m; \xi) \equiv \sum_{n=0}^{\infty} P_n(l \mid m) \xi^n \tag{8}$$

we find that

$$P(l \mid m; \xi) - \xi \sum_{l'} p(l - l') P(l' \mid m; \xi) = \delta_{lm} + \xi \sum_{l'} q(l, l') P(l' \mid m; \xi) \tag{9}$$

The first term on the right-hand side arises from the initial condition (2). We now take a discrete Fourier transform, writing

$$\tilde{P}(\phi \mid m; \xi) \equiv \sum_{l=-\infty}^{\infty} e^{il\phi} P(l \mid m; \xi) \tag{10}$$

and defining the "structure function"

$$\lambda(\phi) \equiv \sum_{l=-\infty}^{\infty} e^{il\phi} p(l) \tag{11}$$

$$= \frac{1}{z} e^{-i\phi} + \left(1 - \frac{1}{z}\right) e^{i\phi} \tag{12}$$

Equation (9) reduces to

$$\tilde{P}(\phi \mid m; \xi) = \frac{e^{im\phi}}{1 - \xi\lambda(\phi)} + \frac{(\xi/z) P(0 \mid m; \xi)(e^{i\phi} - e^{-i\phi})}{1 - \xi\lambda(\phi)} \tag{13}$$

and inverting the discrete Fourier transform we find that

$$P(l \mid m; \xi) = G(l \mid m; \xi) + (\xi/z) P(0 \mid m; \xi) H(l; \xi) \tag{14}$$

where

$$G(l \mid m; \xi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\phi(m-l)}\, d\phi}{1 - \xi\lambda(\phi)} \tag{15}$$

and

$$H(l; \xi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-il\phi}(e^{i\phi} - e^{-i\phi})\, d\phi}{1 - \xi\lambda(\phi)} \tag{16}$$

Setting $l = 0$ in (14) yields an algebraic equation for $P(0 \mid m; \xi)$, so that

$$P(0 \mid m; \xi) = \frac{G(0 \mid m; \xi)}{1 - (\xi/z) H(0; \xi)} \tag{17}$$

and the generating function for $P_n(l \mid m)$ is explicitly determined:

$$P(l \mid m; \xi) = G(l \mid m; \xi) + \frac{G(0 \mid m; \xi) H(l; \xi)}{(z/\xi) - H(0; \xi)} \tag{18}$$

It is a trite exercise in residue calculus to determine $G(l \mid m; \xi)$ and $H(l; \xi)$ in terms of elementary functions. With $t_+(\xi)$, $t_-(\xi)$ defined by

$$t_{\pm}(\xi) = \frac{z \pm \left[z^2 - 4\xi^2(z-1)\right]^{1/2}}{2\xi(z-1)} \tag{19}$$

we find that

$$G(l \mid m; \xi) = \begin{cases} \dfrac{z t_-(\xi)^{m-l}}{\left[z^2 - 4\xi^2(z-1)\right]^{1/2}}, & m \geqslant l \\[2ex] \dfrac{z t_+(\xi)^{m-l}}{\left[z^2 - 4\xi^2(z-1)\right]^{1/2}}, & m \leqslant l \end{cases} \tag{20}$$

and

$$H(l;\xi)=\begin{cases}\dfrac{-z(z-2)t_-(\xi)}{\left[z^2-4\xi^2(z-1)\right]^{1/2}}, & l=0\\[2ex] \dfrac{z(t_+)^{-l}\left\{t_+-(t_+)^{-1}\right\}}{\left[z^2-4\xi^2(z-1)\right]^{1/2}}, & l\geqslant 1\end{cases}\tag{21}$$

Of particular importance in the ensuing analysis is the behavior of $P(0\,|\,m;\xi)$ near $\xi=1$. A little algebra establishes that

$$G(0\,|\,m;\xi)=\frac{z}{(z-2)(z-1)^m}\left\{1-\left[\frac{mz}{z-2}+\frac{4(z-1)}{(z-2)^2}\right](1-\xi)+O([1-\xi]^2)\right\}\tag{22}$$

and

$$H(0;\xi)=-\frac{z}{z-1}\left\{1-\left[\frac{z}{z-2}+\frac{4(z-1)}{(z-2)^2}\right](1-\xi)+O([1-\xi]^2)\right\}\tag{23}$$

so that

$$P(0\,|\,m;\xi)=\left\{\frac{z-1}{z-2}\right\}(z-1)^{-m}\left\{1-\frac{[mz(z-2)+2(z-1)]}{(z-2)^2}(1-\xi)+O([1-\xi]^2)\right\}\tag{24}$$

3. RANDOM WALK STATISTICS

Virtually all random walk statistics characterizing the long-time $(n\to\infty)$ behavior of a Pólya walk on a Bethe lattice can be inferred from the generating function $P(l\,|\,m;\xi)$ for site occupation probabilities, defined and evaluated in Section 2, and a closely related first passage time generating function, which we now introduce. Let $F_n(m)$ denote the probability that a walker whose starting coordinate is $m>0$ arrives at the origin for the first time on the nth step. We define

$$F(m;\xi)=\sum_{n=1}^{\infty}F_n(m)\xi^n\tag{25}$$

and relate $F(m;\xi)$ to $P(l\mid m;\xi)$ using the following argument.[9] Decompose all possible walks starting with coordinate m and terminating after n steps at the origin in terms of first visits to the origin. Then, clearly,

$$P_n(0\mid m)=\delta_{n,0}\delta_{m,0}+\sum_{j=1}^{n}F_j(m)P_{n-j}(0\mid 0) \tag{26}$$

the jth term in the sum accounting for a walk in which the walker first reached the origin after j steps. Multiplying (26) by ξ^n and summing gives

$$P(0\mid m;\xi)=\delta_{m,0}+F(m;\xi)P(0\mid 0;\xi) \tag{27}$$

so that

$$F(m;\xi)=\{P(0\mid m;\xi)-\delta_{m,0}\}/P(0\mid 0;\xi) \tag{28}$$

3.1. Probability of Reaching a Given Site

We calculate the probability $R(m)$ that a walker ever reaches a specified "target" site, m bonds removed from the starting site. In view of the topological equivalence of all sites, there is no loss of generality in selecting the target site as the origin of coordinates. We therefore need only calculate the probability $R(m)$ that a walker with initial coordinate m ever arrives at the origin. Partitioning the probability of ever arriving over the possible arrival times, we have

$$R(m)=\sum_{n=1}^{\infty}F_n(m)=\lim_{\xi\to 1^-}F(m;\xi) \tag{29}$$

We consider first the probability $R(0)$ that the walker ever returns to the starting point. From Eqs. (28) and (29), setting $m=0$, we find

$$R(0)=1/(z-1) \tag{30}$$

In the degenerate case $z=2$ (a linear chain) the walker is certain to return to the starting point (this result is well known[6–8]). For $z\geqslant 3$ there is a finite probability of "escape" from the starting point (failure to ever return), i.e., the walk is *transient*. More generally, for $m\geqslant 1$,

$$R(m)=(z-1)^{-m} \tag{31}$$

a rapidly decaying function of m, so long as $z\geqslant 3$. [For $z=2$, $R(m)=1$ for *all* m.] The result that $R(0)=R(1)$, which follows from (30) and (31), can be derived from first principles: The walker's first step takes him to a nearest-neighbor of the starting point. The probability of return to the starting point is then exactly the probability that a walker ever visits a specified nearest neighbor of the starting point. An analog of this result holds for a Pólya walk on any lattice for which all sites are topologically equivalent.

3.2. Mean First Passage Times

When $z = 2$ (a linear chain), all sites are certain to be visited, and the mean time (step number) for the first return to the origin, or the first visit to any other specified site (usually called the *mean first passage time* to the origin, or the specified site) is infinite.[9] For $z \geqslant 3$, the expected time that an observer at a given lattice site will have to wait before seeing the walker arrive for the first time is infinite, since there is a finite probability $1 - R(m)$ that the observer waits forever. In this sense, the mean first passage time is infinite. It has been noted by Lindenberg *et al.*[10] that the *conditional mean first passage time*, i.e., the first passage time for those walks which do reach the specified site, is of some interest. This quantity is given by

$$\tau_c(m) = R(m)^{-1} \sum_{n=1}^{\infty} nF_n(m) \tag{32}$$

since $F_n(m)/R(m)$ is the conditional probability that the walker first reaches a specified site m bonds removed from his starting point on the nth step, given that he does reach this site. Thus

$$\tau_c(m) = R(m)^{-1} \frac{\partial}{\partial \xi} F(m;\xi)\bigg|_{\xi=1^-} \tag{33}$$

$$= \begin{cases} \dfrac{2(z-1)}{(z-2)}, & m = 0 \\ \dfrac{mz}{z-2}, & m > 0 \end{cases} \tag{34}$$

The conditional mean first passage time to any site is finite when $z \geqslant 3$, a result which we may interpret physically by saying that if a walk endures for more than a small number of steps, it is nearly certain that the walker will be far from the origin by this time, and so highly unlikely to return. The result that $\tau_c(0) = 1 + \tau_c(1)$ can be derived from first principles [using the same argument as that outlined above to show that $R(0) = R(1)$]. We note that

$$\tau_c(0) \to 2 \quad \text{and} \quad \tau_c(m) \to m \qquad (m \geqslant 1) \tag{35}$$

as $z \to \infty$, so that for a Bethe lattice with a very high coordination number, a walker following anything but the most direct path to a target site is unlikely to ever arrive there.

3.3. Expected Number of Distinct Sites Visited

Montroll and Weiss[11] have shown how the large n asymptotic form of S_n, the expected number of distinct sites visited in an n-step walk, can be determined for any Pólya walk on a Bravais lattice. Their analysis adapts quite readily to the Bethe lattice, and one can show that

$$S(\xi) \equiv \sum_{n=0}^{\infty} S_n \xi^n = (1-\xi)^{-2} P(0 \mid 0; \xi)^{-1} \tag{36}$$

Hence, using a Tauberian theorem,[11] we find that as $n \to \infty$,

$$S_n \sim n/P(0 \mid 0; 1) = \left\{ \frac{z-2}{z-1} \right\} n \tag{37}$$

provided that $z \geqslant 3$. The case $z = 2$ (a linear chain) has been analyzed by Montroll and Weiss[11] and other authors,[12] who show that

$$S_n \sim (8n/\pi)^{1/2} \tag{38}$$

Montroll and Weiss[11] have also shown how the statistics for repeated occupancy of a given site can be analyzed. Using their techniques one can easily show that the expected number of distinct sites visited *at least* r times ($S_n^{(r)}$) and the expected number of distinct sites visited *exactly* r times ($V_n^{(r)}$) have the asymptotic forms

$$S_n^{(r)} \sim \frac{(z-2)n}{(z-1)^r} \tag{39}$$

$$V_n^{(r)} \sim \frac{(z-2)^2 n}{(z-1)^{r+1}} \tag{40}$$

for a Bethe lattice with $z \geqslant 3$.

4. WALK ON A BETHE LATTICE IN A CENTRAL FIELD

We have shown how Fourier transform and generating function techniques may be used to analyze random walks on a translationally invariant Bethe lattice, i.e., one for which all sites carry equivalent transition probabilities. If the transition probabilities vary from site to site, the problem becomes extremely difficult, and experience with Bravais lattices leads one to conclude that any general analysis is impossible. However, in special cases the problem can be solved, and we proceed to give one example. (The possibility of solving this particular problem was suggested to the authors by an elegant paper of Gillis,[13] who solved its one-dimensional analog.

Gillis' problem and the present problem are solvable for the same reason: a first-order ordinary differential equation for the Fourier transformed generating function can be derived.)

With coordinates defined as in earlier sections, we consider a walk starting m sites from the origin, with transition probability law

$$\gamma(l,l') = \begin{cases} \delta_{l,l'+1}, & l'=0 \\ \left[1 - \frac{1}{z} - \frac{\kappa}{l'}\right]\delta_{l,l'+1} + \left[\frac{1}{z} + \frac{\kappa}{l'}\right]\delta_{l,l'-1}, & l' \neq 0 \end{cases} \tag{41}$$

where

$$|\kappa| < \min(1/z, 1 - 1/z) \tag{42}$$

If $\kappa > 0$, we have an "attractive potential" (that is, a bias towards the origin), which decays with distance from the origin, while if $\kappa < 0$, there is repulsion from the origin (i.e., bias away from the origin). As in Section 2 we map the problem onto a biased one-dimensional random walk, with the defect at the origin now supplemented by an additional, site-dependent bias. We write

$$\gamma(l,l') = p(l-l') + q(l,l') \tag{43}$$

where $p(l) = (1 - 1/z)\delta_{l,1} + (1/z)\delta_{l,-1}$ as before, and

$$q(l,l') = \begin{cases} \frac{1}{z}\delta_{l,1} - \frac{1}{z}\delta_{l,-1}, & l'=0 \\ -\frac{\kappa}{|l'|}\operatorname{sgn}(l')\delta_{l,l'+1} + \frac{\kappa}{|l'|}\operatorname{sgn}(l')\delta_{l,l'-1}, & l' \neq 0 \end{cases} \tag{44}$$

where $\operatorname{sgn}(l')$ denotes the sign of l'. Introducing the generating function $P(l\,|\,m;\xi)$ as in Section 2, we find in place of Eq. (13) the more complicated equation

$$\begin{aligned}\tilde{P}(\phi\,|\,m;\xi) = {} & \frac{e^{im\phi}}{1-\xi\lambda(\phi)} + \frac{(\xi/z)P(0\,|\,m;\xi)(e^{i\phi}-e^{-i\phi})}{1-\xi\lambda(\phi)} \\ & - \kappa\xi \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{\operatorname{sgn}(l)}{|l|} e^{il\phi} \frac{(e^{i\phi}-e^{-i\phi})}{1-\xi\lambda(\phi)} P(l\,|\,m;\xi)\end{aligned} \tag{45}$$

We recall that

$$P(l\,|\,m;\xi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\theta l}\tilde{P}(\theta\,|\,m;\xi)\,d\theta \tag{46}$$

Substitution of this expression into the right-hand side of (45) and interchanging orders of integration and summation (an operation rigorously justifiable through the theory of generalized functions[14]), we find an integral equation for $\tilde{P}(\phi \mid m; \xi)$:

$$\tilde{P}(\phi \mid m; \xi) = \frac{e^{im\phi}}{1 - \xi\lambda(\phi)} + \frac{(2i\xi/z)P(0 \mid m; \xi)}{1 - \xi\lambda(\phi)} \sin\phi$$

$$+ \frac{2\kappa\xi \sin\phi}{\pi[1 - \xi\lambda(\phi)]} \int_{-\pi}^{\pi} \sum_{l=1}^{\infty} \frac{\sin\{l(\phi - \theta)\}}{l} \tilde{P}(\theta \mid m; \xi)\, d\theta \tag{47}$$

Since[15]

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{1}{2}\{\pi \operatorname{sgn}(x) - x\} \tag{48}$$

for $-2\pi < x < 2\pi$, and $(d/d\phi)\operatorname{sgn}(\phi - \theta) = 2\delta(\phi - \theta)$, multiplying (47) by $[1 - \xi\lambda(\phi)]/\sin\phi$ and differentiating with respect to ϕ gives a first-order differential equation for $\tilde{P}(\phi \mid m; \xi)$, which can be written in the form

$$\frac{d\tilde{P}}{d\phi} + \left\{ \frac{d}{d\phi} \ln\left[\frac{1 - \xi\lambda(\phi)}{\sin\phi} \right] - \frac{2\kappa\xi \sin\phi}{1 - \xi\lambda(\phi)} \right\} \tilde{P}$$

$$= \left\{ -2\kappa\xi P(0 \mid m; \xi) + \frac{d}{d\phi}(e^{im\phi}/\sin\phi) \right\} \frac{\sin\phi}{1 - \xi\lambda(\phi)} \tag{49}$$

and solved using the integrating factor

$$\frac{1 - \xi\lambda(\phi)}{\sin\phi} \exp\left[-2\kappa\xi \int_{-\pi}^{\phi} \frac{\sin\nu\, d\nu}{1 - \xi\lambda(\nu)} \right]$$

We find, after a little algebra, that

$$\tilde{P}(\phi \mid m; \xi) = \frac{e^{im\phi}}{1 - \xi\lambda(\phi)} + \frac{A \sin\phi}{1 - \xi\lambda(\phi)} \exp\left[2\kappa\xi \int_{-\pi}^{\phi} \frac{\sin\nu\, d\nu}{1 - \xi\lambda(\nu)} \right] + \frac{2\kappa\xi \sin\phi}{1 - \xi\lambda(\phi)}$$

$$\times \left\{ \int_{-\pi}^{\phi} \frac{e^{im\theta}}{1 - \xi\lambda(\theta)} \exp\left[2\kappa\xi \int_{\theta}^{\phi} \frac{\sin\nu\, d\nu}{1 - \xi\lambda(\nu)} \right] d\theta \right.$$

$$\left. - P(0 \mid m; \xi) \int_{-\pi}^{\phi} \exp\left[2\kappa\xi \int_{\theta}^{\phi} \frac{\sin\nu\, d\nu}{1 - \xi\lambda(\nu)} \right] d\theta \right\} \tag{50}$$

where A is a constant of integration. As we now show, two simultaneous linear equations relating A and $P(0 \mid m; \xi)$ can be derived. First, we integrate (50) with respect to ϕ from $-\pi$ to π and use (46) to deduce that

$$A\left\{\exp\left[2\kappa\xi\int_{-\pi}^{\pi}\frac{\sin\nu\,d\nu}{1-\xi\lambda(\nu)}\right]-1\right\}$$

$$=-2\kappa\xi\int_{-\pi}^{\pi}\frac{e^{im\theta}}{1-\xi\lambda(\theta)}\exp\left[2\kappa\xi\int_{\theta}^{\pi}\frac{\sin\nu\,d\nu}{1-\xi\lambda(\nu)}\right]d\theta$$

$$+2\kappa\xi P(0\,|\,m;\xi)\int_{-\pi}^{\pi}\exp\left[2\kappa\xi\int_{\theta}^{\pi}\frac{\sin\nu d\nu}{1-\xi\lambda(\nu)}\right]d\theta \tag{51}$$

In the degenerate case of a linear chain [where $z=2$ and $\lambda(\nu)=\cos\nu$], the coefficient of A in (51) vanishes, and $P(0\,|\,m;\xi)$ is determined uniquely without the need to find A:

$$P(0\,|\,m;\xi)=\frac{\int_{-\pi}^{\pi}(e^{im\theta}/1-\xi\lambda(\theta))\exp\left[2\kappa\xi\int_{\theta}^{\pi}(\sin\nu\,d\nu/1-\xi\lambda(\nu))\right]d\theta}{\int_{-\pi}^{\pi}\exp\left[2\kappa\xi\int_{\theta}^{\pi}(\sin\nu\,d\nu/1-\xi\lambda(\nu))\right]d\theta} \tag{52}$$

The integrals in (52) can be expressed in terms of the hypergeometric function and the gamma function:

$$P(0\,|\,m;\xi)=\frac{\xi^m\Gamma(m+1+2\kappa)\,{}_2F_1(m/2+1/2+\kappa,m/2+1\kappa;m+1;\xi^2)}{2^m m!\,\Gamma(1+2\kappa)\,{}_2F_1(\kappa+1/2,\kappa;1;\xi^2)} \tag{53}$$

When $m=0$, we recover the result of Gillis.[13] [For the case $z=2$, the fact that the origin is a reflecting barrier does not affect the calculation of $P(0\,|\,m;\xi)$. Using properties of the gamma and hypergeometric functions, one can show that (53) is invariant under the replacement of m by $-m$.]

When $z\geqslant 3$, it is necessary to return to the integral equation to obtain a second relation between A and $P(0\,|\,m;\xi)$. The resulting algebra can be reduced somewhat by noting that if (50) is subtracted from (47), cancellation of the common factor $\sin\phi/[1-\xi\lambda(\phi)]$ and evaluation at $\phi=-\pi$ yields the equation

$$(2i\xi/z)P(0\,|\,m;\xi)+\frac{\kappa\xi}{\pi}\int_{-\pi}^{\pi}\theta\tilde{P}(\theta\,|\,m;\xi)\,d\theta=A \tag{54}$$

Substitution of the solution of the differential equation [given by (50)] into (54) yields the required linear relation between A and $P(0\,|\,m;\xi)$. Although the solution is now completely determined, the resulting expressions are rather cumbersome, and we do not exhibit them here.

5. THE SUBJECTIVE DIMENSION OF A BETHE LATTICE

We have shown in the present paper that many of the properties of random walks are as easily deduced for the Bethe lattice as for Bravais lattices. Our ability to solve these problems stems from the observation that a random walk on a Bethe lattice can be mapped onto a biased, one-

Table I. Random Walk Statistics for Various Lattices[a]

Lattice		$R(0)$	$\tau_c(0)$	$S_n\ (n \gg 1)$
One-dimensional biased[b] $\gamma(l, l') = (1 - 1/z)\delta_{l,l'+1} + (1/z)\delta_{l,l'-1}$		$\frac{2}{z}$	$\frac{2(z-1)}{(z-2)}$	$\frac{(z-2)}{z}n$
Bethe lattice[c] with coordination number z		$\frac{1}{z-1}$	$\frac{2(z-1)}{(z-2)}$	$\left\{\frac{z-2}{z-1}\right\}n$
	$d = 1$	1	∞	$\left\{\frac{8n}{\pi}\right\}^{1/2}$
	$d = 2$	1	∞	$\frac{\pi n}{\ln n}$
Cubic lattice[d] of dimension d	$d = 3$[e]	$1 - W(3)^{-1}$	∞	$n/W(3)$
	$d = 4$	$1 - W(4)^{-1}$	∞	$n/W(4)$
	$d > 4$	$1 - W(d)^{-1}$	finite	$n/W(d)$
	$d \gg 4$	$\sim \frac{1}{2d}$	~ 2	$\sim n$

[a]Notation:

$R(0)$ = probability of return to the starting point

$\tau_c(0)$ = mean time to return to the starting point, given that return occurs

S_n = mean number of distinct sites visited in a walk of n steps

$$W(d) = \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi} d\theta_1 \ldots \int_{-\pi}^{\pi} d\theta_d [1 - (\cos\theta_1 + \cdots + \cos\theta_d)]^{-1}$$

$$= 1 + \frac{1}{2d} + O(d^{(-2)}), \qquad d \to \infty$$

[b]Easily calculated using generating function techniques (Ref. 7) or more elementary means (Ref. 9).

[c]Results of the present paper.

[d]$R(0)$ from Póyla (Ref. 8) and Montroll (Ref. 6); $\tau_c(0)$ from Lindenberg *et al.* (Ref. 10); S_n from Montroll and Weiss (Ref. 11).

[e]The exact value can be expressed in terms of gamma functions

$$W(3) = \frac{\sqrt{6}}{288\pi^3}\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{5}{24}\right)\Gamma\left(\frac{7}{24}\right)\Gamma\left(\frac{11}{24}\right) \simeq 0.34.$$

[M. L. Glasser, personal communication 1982, based on a corrected version of Glasser and Zucker (Ref. 16).]

dimensional lattice walk with a single defect, at the cost of losing all information about lattice sites, other than their distance from a single, specified site (the origin of coordinates). In view of this mapping, one may choose to regard the Bethe lattice as a kind of biased, one-dimensional system, and the Bethe lattice is certainly one-dimensional in the sense that any two sites are connected by a unique linear chain. However, the opinion has often been advanced that the Bethe lattice is essentially an infinite dimensional system. This opinion, together with the observation that many statistical or cooperative phenomena are qualitatively independent of dimension above a certain critical dimension, has led to the use of the Bethe lattice as a simple analytic probe of the properties of Bravais lattices of sufficiently large dimensionality.

It is interesting to see what light random walk statistics can shed upon the "dimensionality" of the Bethe lattice. We give in Table I a variety of random walk statistics for Pólya's walk on cubic lattices of arbitrary dimension d, Pólya's walk on a Bethe lattice of coordination number z, and a biased one-dimensional walk with transition probability

$$\gamma(l, l') = \left(1 - \frac{1}{z}\right)\delta_{l,l'+1} + \frac{1}{z}\delta_{l,l'-1}$$

(i.e., one which is formally identical to Pólya's walk on a Bethe lattice of coordination number z, except for the omission of the defect at $l' = 0$). It can be seen from Table I that the Pólya walk on a Bethe lattice does have some qualitative similarities to a Pólya walk on a cubic lattice of dimension $d > 4$. However, the correspondence between the Bethe lattice and a biased linear chain is just as good. In any event, even if one agrees that for the random walk problems considered here, a Bethe lattice is equivalent to a simple cubic lattice of dimension greater than 4, it is most certainly inappropriate to say that a Bethe lattice of finite coordination number z is equivalent to an infinite-dimensional cubic lattice, since the latter gives $R(0) = 0$, while the former gives $R(0) > 0$. The authors suggest that great caution should be exercised in assigning any "dimension" to a Bethe lattice of coordination number z, and in carrying over Bethe lattice results to Bravais lattices.

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NOTE ADDED IN PROOF

It has recently been noted by P. W. Kasteleyn (preprint, 1982) that Pólya's walk on a Bethe lattice with even coordination number z is equivalent to a random walk on a group considered by H. Kesten [*Trans. Amer. Math. Soc.* **92**:336 (1959)]. Using Kesten's solution, Kasteleyn deduces several random walk statistics. His results for even z agree with the formulae derived in Section 3 for all integers $z > 2$.

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